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THE
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On some Questions of Combination. *By* PROFESSOR DE MORGAN.

QUESTIONS of combination are so connected with an actuary's business, that no apology is necessary for offering solutions to the notice of the readers of this *Journal.*

When two systems, A and B, of combinations, each having its own individual cases, are so related that each case of A admits of being converted into a case of B, and into one only, by a rule which will not convert any other case of A into that same case of B, it may be said that the system A is *convertibly* within B. It is obvious that if A be convertibly within B, and B convertibly within A, the numbers of cases in the systems A and B are equal.

Let any system of combinations in each case of which m things are selected out of n be denoted by (m, n), which is rather a grammatical than an algebraical abbreviation. The n things combined may be the n numbers $1, 2, 3, \ldots n$. When (m, n) consists of simple combinations without repetition, let their number be denoted by m_n, which therefore represents $n(n-1) \ldots (n-m+1)$ divided by $1 . 2 . 3 \ldots m$. The obvious theorem $m_n = (n-m)_n$ is often used. In writing cases of combination, independent of permutation, a prescribed order is desirable, which may be the numerical, the alphabetical, or any other.

Any combination of m numbers has $m-1$ intervals, and each interval marks either a *break* or a *sequence.* Thus in 4 7 8 9 15 16 we see a break of two, two sequences, a break of five, and a sequence.

Any arrangement or disposition may be converted into any other by a succession of interchanges of contiguous components. Thus ABCD can be converted into CDBA by successively becoming BACD, BCAD, BCDA, CBDA, CDBA; each change being only interchange of neighbouring letters.

Required (m, n), so that in each combination there shall be a break of p or more in every interval. Take $(5, 25)$ and $p=3$ to reason on. Add $p(m-1)$, or 12, to n, or 25, and let A be the system $1, 2, 3 \ldots 25$, and let B be the system $1, 2, 3 \ldots 37$. Then $(5, 25)$ simply, and $(5, 37)$ with breaks of p or more in all the intervals, are each convertibly within the other. Thus—

Case of A . .	2	3	10	12	13	4	5	6	11	12	21	22	23	24	25
Add	0	3	6	9	12	0	3	6	9	12	0	3	6	9	12
Case of B . .	2	6	16	21	25	4	8	12	20	24	21	25	29	33	37
Case of B . .	1	8	16	25	33	7	11	15	19	23	18	23	27	31	37
Subtract . .	0	3	6	9	12	0	3	6	9	12	0	3	6	9	12
Case of A . .	1	5	10	16	21	7	8	9	10	11	18	20	21	22	25

By turning the A case *abcde* into $a(b+3)(c+6)(d+9)(e+12)$, we get a B case, obtainable by this rule in no other way; and, *vice versâ*, by turning the B case *abcde* into $a(b-3)(c-6)(d-9)(e-12)$. Hence, number in (m, n) with breaks of p or more $= m_{n-(m-1)p}$.

The number in (m, n) without any sequences $(p=1)$ is m_{n-m+1}. Thus $(3, 6)$, without sequences, gives 3_4 or 4 cases: they are, 1 3 5, 1 3 6, 1 4 6, 2 4 6.

The system (m, n) with breaks in *specified* intervals, is of the same number of cases, whatever the specified intervals may be. Take eight with breaks in the third, fifth, and sixth intervals, of which a case is 1 2 3 | 7 8 | 10 | 12 13. There is but one way of removing a break into a contiguous interval. Thus the first break can only be thrown forward by conversion of the case into 1 2 3 4 | 8 | 10 | 12 13, and backward only by conversion into 1 2 | 6 7 8 | 10 | 12 13. Hence two systems (m, n) with k breaks in intervals which have only one difference, and that difference affecting two contiguous intervals, are of the same number of cases. Hence, as before shown, the same is true if the k breaks be in any two sets of assigned intervals.

Required the number in (m, n) with k breaks of p or more in certain assigned intervals, and no other breaks. Let the first k intervals be taken. Then we have $k+1$ numbers without a sequence, followed by $m-k-1$ numbers with nothing but sequences, the $(k+1)$th and $(k+2)$th being also in sequence. The $(k+1)$th number cannot be higher than $n-(m-k-1)$; and, the $(k+1)$th being known, all the rest are determined. Hence we have merely to determine $(k+1, n-(m-k-1))$ with nothing but breaks of p or more. This, as shown, is $(k+1)_{n-(m-k-1)-kp}$, or $(k+1)_{n-m-(p-1)k+1}$. Thus (5, 15), with the first three intervals breaks of three or more ($m=5$, $n=15$, $k=3$, $p=3$), gives 4_5 or 5 cases. These cases are 1 5 9 13 14, 1 5 9 14 15, 1 5 10 14 15, 1 6 10 14 15, 2 6 10 14 15.

Required the number in (m, n) with k breaks of p or more in *any* intervals, and no other breaks. The number of ways of selecting k out of the intervals is k_{m-1}, the number belonging to the question for each such selection is as above; whence the answer is $k_{m-1} \times (k+1)_{n-m-(p-1)k+1}$.

If we make no limitation as to the character of the breaks, then $p=1$, and the number in (m, n) with breaks in k assigned intervals is $(k+1)_{n-m+1}$; while the number of cases with k breaks in any intervals is $k_{m-1} \times (k+1)_{n-m+1}$.

This last case may be simply verified. Since the cases for all values of k give the whole number of combinations, we have $m_n = 0_{n-1} . 1_{n-m+1} + 1_{m-1} . 2_{n-m+1} + 2_{m-1} . 3_{n-m+1} + \ldots$ The second side is the coefficient of x^{-1} in the product of $0_{m-1} + 1_{m-1} . x + \ldots$ and $0_{n-m+1} + 1_{n-m+1} . x^{-1} + \ldots$ These factors are $(1+x)^{m-1}$ and $(1+x^{-1})^{n-m+1}$, and their product is $(1+x)^n : x^{n-m+1}$. The coefficient of x^{-1} in this is that of x^{n-m} in $(1+x)^n$; which is $(n-m)_n$, or m_n, the first side.

I shall now consider the solution of another class of problems, not, as will appear, essentially different from the first class. The system (m, n), when any attainable amount of repetition is allowed, has a number of cases which is obtained by altering the negative signs of m_n into positive ones. Thus $(4, n)$, allowing repetition, contains $n(n+1)(n+2)(n+3) : 1 . 2 . 3 . 4$ cases. That is, (m, n) with repetition has m_{m+n-1} cases. The following is, I think, as easy a *direct* proof as this problem can admit of:—

Let the case be (12, 20) with repetition, of which I am to show that the system is convertibly within (12, 31) without repetition, and the converse.

Let system A be formed out of 20 letters, *a, b, c,* &c., with parcels of each for repetition. Let B be formed out of one of each of the 20 letters *a, b, c,* &c., and the numbers 1, 2, 3 . . . 11. Each case in either system contains 12: each A case, all letters; each B case, letters, or letters and numbers.

First, A is convertibly within B. Take any case of A: if without repetition, it is itself a case of B; if with repetition, let it be $bbddddfgggll$. Over all but the first letter write $1, 2, 3 \dots 11$. This gives

$$\begin{array}{cccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ b & b & d & d & d & d & f & g & g & g & l & l \end{array}$$

Now discard every repetition, and put the number written over the discarded letter in its place. This gives $b\,1\,d\,3\,4\,5\,f\,g\,8\,9\,l\,11$, a case of B, the only one thus producible from the case of A, and not producible from any other.

Secondly, B is convertibly within A. Take any case of B: if without numbers, it is a case of A; if with numbers, let it be $a\,b\,m\,n\,p\,2\,5\,6\,7\,9\,10\,11$. Begin writing 1, 2, 3, &c., over the letters; but as soon as we thus come to the first number in the case, write it *under* the letter, and continue writing under that same letter as long as the numbers of our case are in sequence. Write the first number we come to which is not in our case *over* the letter in use, and go on writing over the letters until we come to the next number which is in the case; and so on.

$$\begin{array}{ccccc} 1 & 3 & 4 & 8 & \\ a & b & m & n & p \\ & 2 & & 5 & 9 \\ & & & 6 & 10 \\ & & & 7 & 11 \end{array}$$

Now discard the numbers written over the letters, and for each number under any letter substitute that letter. This gives $abbmnnnnpppp$, a case of A, the only one producible from the given case of B, and not producible by this rule from any other. Hence each system is convertibly within the other, and the numbers of cases in the two are equal.

The two rules are real inverses of each other. Thus $abbmnnnnpppp$ reverts, by the first rule, into $a\,b\,2\,m\,n\,5\,6\,7\,p\,9\,10\,11$, from which it was obtained by the second rule. Nor can the numbers in the second rule ever *outrun* the letters. With $abmnp$, the latest possible number is 5: had the numbers been $5\,6 \dots 11$, all would have come under p; and any alteration of numbers throws us backward, none forward.

In the system with repetition, m may exceed n: thus (12, 3) with repetition is (12, 14) without repetition, or 12_{14}.

Every problem involving *partition with arrangement* is also a problem of *distribution of undistinguishables*, and a problem of *combination of distinguishables*. For instance, the common permutation problem of the arrangements of the six numbers, a, b, c, d, e, f, may be represented in either of the following ways:—

Partition, &c. In how many ways can $a+b+\ldots+f$ be divided into the six parts, $a, b, \ldots f$?

Distribution, &c. In how many ways can $a+b+\ldots+f$ undistinguishable counters be distributed among six boxes, so that one box shall receive a, another b, &c.?

Combination, &c. Let every box be marked with a letter, P, Q, &c., and in every case of the last let each counter take the letter of the box it is in. Either of the preceding problems then is the following:—How many combinations exist of the letters P, Q, R, S, T, U, the condition being that some one of these letters shall occur a times, some other b times, &c.?

The answer in each of these three cases is the product $1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6$.

Now let it be asked, In how many arrangements the number m may be divided into n numbers? Write down m units, with $m-1$ intervals. To divide this into n lots in every possible way, we must insert in every possible way $n-1$ marks of partition, not more than one in any interval. The number of ways of doing this is $(n-1)_{m-1}$, which is the answer required. It also solves the following questions—In how many ways can m undistinguishable counters be placed in n boxes, one or more in each box? How many combinations are there of m letters out of n, allowing repetition, and no letter being entirely absent?

Since in the preceding there must be one unit in every place, deduct n units, place them, and distribute the rest: the answer then to the question, in how many ways $m-n$ can be arranged into n numbers *or zeros*, is $(n-1)_{m-1}$. Hence m can be divided into n numbers or zeros in $(n-1)_{m+n-1}$ or m_{m+n-1} ways. And this is also the number of ways in which m undistinguishable counters can be placed in n boxes, so that any boxes may be left empty: and the number of ways in which m combinations can be taken out of n letters, with any repetition; as before shown. This is the most simple proof of the formula for combinations with repetition.

In (m, n) with repetition, supposing m ⁊ or $=pn$, in how many combinations does each of the n letters occur at least p times? This obviously asks the number, still with repetition, in $(m-pn, n)$; and the answer is $(m-pn)_{m-pn+n-1}$.

Again; in how many combinations of m letters do precisely l letters occur, each letter occurring at least p times? Choose certain l letters: the number of ways with these letters is

$(m-pl)_{m-pl+l-1}$. But the choice of l letters can be made in l_n ways: hence $l_n \times (m-pl)_{m-pl+l-1}$ is the number required.

This is the problem of breaks in another form. To make this clearer, let us take a more extensive B case to be converted into one of A. Let the system be ·(50, 80) with repetition, and let the B case have 16 letters and 34 of the 49 numbers. Let it be $a, b, \ldots p, 1, 2, \ldots 8 | 12 | 16, 17, \ldots 22 | 27, 28 | 31, 32, \ldots 42 | 44 | 47, 48, 49$. We have then

9	10	11	13	14	15	23	24	25	26	29	30	43	45	46	
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
1			12			16				27		31	44		47
2						17				28		32			48
⋮						⋮						⋮			49
8						22						42			

$$a^9\, b\, c\, d^2\, e\, f\, g^8\, h\, i\, j\, k^3\, l\, m^{13}\, n^2\, o\, p^4$$

using a^9 in abbreviation of nine a's, &c. Every break in the numbers carries us to a new letter; and having *six* breaks in the B case chosen, we have *seven* letters which undergo repetition in the A case deduced. So many ways, then, as we can take 34 out of 49 numbers with six breaks, in so many ways can we take combinations of 41 out of 16 letters, each combination having seven letters, all of which are repeated two or more times. Generally, in so many ways as we can take m out of n numbers with k breaks, in so many ways can we take $m+k+1$ out of $n-m+1$, each combination having $k+1$ letters, and each letter repeated at least twice. This latter, by the last problem, can be done in $(k+1)_{n-m+1} \times (m+k+1-2k-2)_{m+k+1-2k-2+k+1-1}$ ways, or $(k+1)_{n-m+1} \times (m-k-1)_{m-1}$, or $k_{m-1} \times (k+1)_{n-m+1}$ ways. And this is the result already obtained in the former problem.

This connection between problems involving combination with breaks and problems involving combination with repetitions might be carried much further, but what is here given will be sufficient to indicate the method.

There is no distinction more marked than that which exists between problems of partition in which different orders count as different ways, and problems in which all the arrangements of one partition are supposed undistinguishable. We may call these last *ordinate* partitions, choosing the order of magnitude as that in which to write down the parts. Thus the ordinate partitions of 5 into three numbers are only 3 1 1, 2 2 1; while among the inordinate partitions are also found 1, 3, 1, 1, 2, 2, &c.

I treated the question of ordinate partition of simple numbers in the *Cambridge Math. Jo.*, vol. iv., p. 87, carrying the results up to the ordinate partition of the number x into *four*. Sir John Herschel subsequently turned his attention to the same subject, in a paper in the *Phil. Trans.* for 1850. This paper contains a much fuller discussion of the question (in the proportion of 24 pages quarto to three pages octavo), and a complete exhibition of the form of the final result, and the general formula for partition into five. To give the actuary some idea of his good fortune in having only to deal with questions connected with *inordinate* partition, I will copy my own result for ordinate quadripartition. Let β and γ stand for $\frac{1}{2}(-1+\sqrt{-3})$ and $\frac{1}{2}(-1-\sqrt{-3})$: the number of ordinate partitions of x into four is the 864th part of

$$6x^3+18x^2-27x-39+(27x+27)(-1)^x+32(\beta^{x-1}+\gamma^{x-1}-\beta^x-\gamma^x)$$
$$+54\{(\sqrt{-1})^x+(-\sqrt{-1})^x\}$$

No pure algebraical expression of a more simple character can be substituted for the above, but an arithmetical description of an easier working character can be given. The last of the following results is altered from Sir John Herschel's paper. The number of ordinate partitions of x is as follows:—

Two parts. The highest integer which does not exceed $\frac{1}{2}x$.

Three parts. The integer *nearest* (above or below) to $\frac{x^2}{12}$.

Four parts. The integer nearest to $\frac{x^3+3x^2}{144}$ or $\frac{x^3+3x^2-9x}{144}$, according as x is even or odd.

Five parts. The integer nearest to $\frac{x^4+10x^3+10x^2-120x}{2880}$ or $\frac{x^4+10x^3+10x^2-30x}{2880}$, according as x is even or odd.

This last is also the solution of the following problem:—How many combinations are there of x letters out of 5; each letter occurring once at least, and no one occurring oftener than the letter preceding?